

The nonlocal Darboux transformation of the 2D stationary Schrödinger equation and its relation to the Moutard transformation

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Abstract

The nonlocal Darboux transformation of the two - dimensional stationary Schrödinger equation is considered and its relation to the Moutard transformation is established. It is shown that a particular case of the nonlocal Darboux transformation provides the Moutard transformation. New examples of exactly solvable two - dimensional stationary Schrödinger operators with smooth potentials are obtained as an application of the nonlocal Darboux transformation.

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1 Introduction

The two - dimensional stationary Schrödinger equation is a model of physical phenomenon in nonrelativistic quantum mechanics [1], acoustics [2] and tomography [3]. The case of fixed energy for two - dimensional Schrödinger equation is of interest for the theory of two - dimensional integrable nonlinear systems [4], [5] and for the multidimensional inverse scattering theory [6] .

Consider the two - dimensional stationary Schrödinger equation in the form

$$(\Delta - u(x, y)) Y(x, y) = 0 \quad (1)$$

where Δ is the two - dimensional Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The Schrödinger equation is related with the Fokker-Planck equation

$$W_{xx} + W_{yy} + \frac{\partial}{\partial x} (2 h_x W) + \frac{\partial}{\partial y} (2 h_y W) = 0 \quad (2)$$

by substitution [7]

$$Y(x, y) = W(x, y) e^{h(x, y)} \quad (3)$$

if condition

$$u = -\Delta h + h_x^2 + h_y^2 \quad (4)$$

holds.

The Fokker-Planck equation (2) has the conservation law form that yields a pair of potential equations

$$W_x + 2 h_x W - Q_y = 0 \quad (5)$$

$$W_y + 2 h_y W + Q_x = 0 \quad (6)$$

In the paper [8] the special form of Darboux transformation for the potential equations (5), (6) was considered. As the potential variable is a nonlocal variable for the Schrödinger equation that provides the nonlocal Darboux transformation for the Schrödinger equation. It was shown that this nonlocal transformation is a useful tool for obtaining exactly solvable two - dimensional stationary Schrödinger operators. The consideration in the paper [8] is restricted by the simple case $h = 0$. In the present paper the case of arbitrary h is considered and relation of the nonlocal Darboux transformation with the Moutard [9] transformation is established.

2 The nonlocal Darboux transformation and its relation with the Moutard transformation

For completeness we outline some results of the paper [8]. Let us consider linear operator corresponding to the system of equations (5), (6)

$$\hat{L}(h(x, y)) \mathbf{F} = \begin{pmatrix} 2 h_x + \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ 2 h_y + \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

Consider Darboux transformation in the form

$$\hat{L}_D \mathbf{F} = \begin{pmatrix} r_{11} - a_{11} \frac{\partial}{\partial x} - b_{11} \frac{\partial}{\partial y} & r_{12} - a_{12} \frac{\partial}{\partial x} - b_{12} \frac{\partial}{\partial y} \\ r_{21} - a_{21} \frac{\partial}{\partial x} - b_{21} \frac{\partial}{\partial y} & r_{22} - a_{22} \frac{\partial}{\partial x} - b_{22} \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

If linear operators \hat{L} and \hat{L}_D hold the intertwining relation

$$\left(\hat{L}(h(x, y) + s(x, y)) \hat{L}_D - \hat{L}_D \hat{L}(h(x, y)) \right) \mathbf{F} = 0 \quad (7)$$

for any $\mathbf{F} \in \mathcal{F} \supset \text{Ker}(\hat{L}(h))$ where $\text{Ker}(\hat{L}(h)) = \{\mathbf{F} : \hat{L}(h) \mathbf{F} = 0\}$, then for any $\mathbf{F}_s \in \text{Ker}(\hat{L}(h))$ the function $\tilde{\mathbf{F}}(x, y) = \hat{L}_D \mathbf{F}_s(x, y)$ is a solution of the equation $\hat{L}(\tilde{h}) \tilde{\mathbf{F}} = 0$ with new potential $\tilde{h} = h + s$.

If one consider equation (7) on the set \mathcal{F} of arbitrary functions, then treating F_1, F_2 and each of its derivatives as independent variables the following equation can be obtained

$$\left(\frac{\partial(h+s)}{\partial x} \right)^2 + \left(\frac{\partial(h+s)}{\partial y} \right)^2 = \left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial y} \right)^2 \quad (8)$$

This is strong limitation for the new potential $\tilde{h} = h + s$.

In the paper [8] it was proposed to consider equation (7) on the following domain:

$$\mathcal{F}_0 = \{ \mathbf{F} : F_{1x} + 2h_x F_1 - F_{2y} = 0 \}$$

Taking into account this dependance of F_1, F_2 derivatives, the equations for $s, r_{ij}, a_{ij}, b_{ij}$ can be obtained.

It should be noted that instead of restriction to solutions of equation (5) domain \mathcal{F}_0 can be defined by restriction to solutions of equation (6). Because the system of equations (5), (6) has x, y interchange with change of Q sign symmetry, one can simply use this symmetry in resulting formulae.

In the paper [8] the case $h = 0$ was considered. Now let us consider the case of arbitrary h .

When solving equation (7) on the domain \mathcal{F}_0 the special situation arise in the case $s = -2h$ that corresponds to $\tilde{h} = -h$. In this case one can obtain from the equation (7) the following Darboux transformation

$$\hat{L}_D = e^{2h(x,y)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (9)$$

By the formula (4) with $\tilde{h} = -h$ we obtain for the new Schrödinger potential

$$\tilde{u}(x, y) = u(x, y) + 2 \Delta h(x, y) \quad (10)$$

Consider

$$Y_h(x, y) = e^{-h(x, y)} \quad (11)$$

According to the formula (4), Y_h is a solution of the Schrödinger equation with potential u .

Then we get another form of the formula (10)

$$\tilde{u}(x, y) = u(x, y) - 2 \Delta \ln(Y_h(x, y)) \quad (12)$$

This formula coincides with the formula of Moutard transformation [9] for the potential of the Schrödinger equation.

From the formula (9) and relation

$$\tilde{Y}(x, y) = \tilde{W}(x, y) e^{\tilde{h}(x, y)} \quad (13)$$

we have for the new solutions of the Fokker-Planck and Schrödinger equations

$$\tilde{W}(x, y) = e^{2h(x, y)} Q(x, y) \quad (14)$$

$$\tilde{Y}(x, y) = e^{h(x, y)} Q(x, y) \quad (15)$$

One can express W, Q, h by equations (3), (14), (11) through Y, \tilde{Y}, Y_h and substitute to the system of equations (5), (6). The result is

$$\frac{\partial}{\partial x} \left(Y_h(x, y) \tilde{Y}(x, y) \right) + (Y_h(x, y))^2 \frac{\partial}{\partial y} \left(\frac{Y(x, y)}{Y_h(x, y)} \right) = 0 \quad (16)$$

$$\frac{\partial}{\partial y} \left(Y_h(x, y) \tilde{Y}(x, y) \right) - (Y_h(x, y))^2 \frac{\partial}{\partial x} \left(\frac{Y(x, y)}{Y_h(x, y)} \right) = 0 \quad (17)$$

These formulae coincide with the formulae of the Moutard transformation for the solution of the Schrödinger equation.

Thus the case $\tilde{h} = -h$ of the nonlocal Darboux transformation provides the Moutard transformation.

The twofold application of the Moutard transformation can be effective for obtaining nonsingular solvable potentials for the Schrödinger equation [10]. Now we give some formulae of the twofold Moutard transformation in the form convenient for later use.

Let us consider a solvable Schrödinger potential u and select two solutions Y_1, Y_2 of the Schrödinger equation with this potential. Then according to (11) we choose $Y_{h1} = Y_1$ and perform the Moutard transformation by the change of h_1 sign. From (15) we have for the result of Y_2 transformation $\tilde{Y}_2 = Q_{12}/Y_{h1}$, where Q_{12} , according to (5), (6), (3) and (11), satisfies the following system of equations:

$$\frac{\partial Y_2}{\partial x} Y_1 - Y_2 \frac{\partial Y_1}{\partial x} - \frac{\partial Q_{12}}{\partial y} = 0, \quad \frac{\partial Y_2}{\partial y} Y_1 - Y_2 \frac{\partial Y_1}{\partial y} + \frac{\partial Q_{12}}{\partial x} = 0 \quad (18)$$

Then we choose $Y_{h2} = \tilde{Y}_2$ and perform the Moutard transformation by the change of h_2 sign. From (4) for $h = -h_2$ taking into account (18) we get

$$\begin{aligned} \tilde{u} = u + 4(Q_{12})^{-1} & \left(\frac{\partial Y_2}{\partial y} \frac{\partial Y_1}{\partial x} - \frac{\partial Y_2}{\partial x} \frac{\partial Y_1}{\partial y} \right) \\ & + 2(Q_{12})^{-2} \left(\left(\frac{\partial Y_2}{\partial x} Y_1 - Y_2 \frac{\partial Y_1}{\partial x} \right)^2 + \left(\frac{\partial Y_2}{\partial y} Y_1 - Y_2 \frac{\partial Y_1}{\partial y} \right)^2 \right) \end{aligned} \quad (19)$$

For nonsingular u, Y_1, Y_2 one can see from this formula that singularity of \tilde{u} can arise from Q_{12} zeros. Note that Q_{12} is defined by (18) to within arbitrary constant. In some cases, this constant allows to make Q_{12} of constant signs.

According to (16), (17) for $Y = Y_h$ one has simple solution $\tilde{Y} = 1/Y_h$. Therefore $\tilde{\tilde{Y}} = 1/Y_{h2} = Y_1/Q_{12}$ is an example of solution for the Schrödinger equation with potential $\tilde{\tilde{u}}$.

Now let us return to the consideration of the equation (7). In the case s not equal to $-2h$ from the equation (7) on the domain \mathcal{F}_0 we obtain the following Darboux transformation

$$\hat{L}_D = e^{-s(x,y)} \begin{pmatrix} R_1 - \frac{\partial}{\partial y} & R_2 \\ -s_x - R_2 & -s_y + R_1 - \frac{\partial}{\partial y} \end{pmatrix} \quad (20)$$

where $R_1 = F_1/F$, $R_2 = F_2/F$

$$\begin{aligned}
F &= 2s_x(s_x + 2h_x) + 2s_y(s_y + 2h_y) \\
F_1 &= (s_x + 2h_x)(-2(s_{xy} + h_{xy}) + (s_y - 2h_y)s_x) \\
&\quad + (s_y + 2h_y)(s_{xx} - s_{yy} - 2h_{yy} + (s_y - 2h_y)s_y) \\
F_2 &= s_x s_{xx} + 2s_y(s_{xy} + h_{xy}) - s_x(s_{yy} + 2h_{yy}) \\
&\quad - (s_x + 2h_x)s_x^2 - (s_y + 2h_y)s_x s_y \quad (21)
\end{aligned}$$

and s satisfies the following system of two nonlinear differential equations:

$$\begin{aligned}
&(s_x^3 + 2s_x^2 h_x + (2s_y h_y + s_y^2)s_x)s_{xxx} \\
&+ ((s_y - 2h_y)s_x^2 + (-4h_y h_x + 2h_x s_y)s_x + s_y^3 - 4h_y^2 s_y)s_{xxy} \\
&+ (s_x^3 + 6s_x^2 h_x + (8h_x^2 + 2s_y h_y + s_y^2)s_x + 8h_x s_y h_y + 4h_x s_y^2)s_{xyy} \\
&+ ((2h_y + s_y)s_x^2 + (4h_y h_x + 2h_x s_y)s_x + 4h_y s_y^2 + s_y^3 + 4h_y^2 s_y)s_{yyy} \\
&+ (2s_y h_y + s_y^2 - s_x^2)s_{xx}^2 + 2((-2s_y + h_y)s_x + 2h_y h_x - h_x s_y)s_{xx} s_{xy} \\
&+ 2(-h_x s_x + 2h_y^2 + s_y h_y)s_{xx} s_{yy} + 4(s_y h_y - h_x s_x - 2h_x^2)s_{xy}^2 \\
&+ 2(-h_{xx} s_x^2 + 2(-h_{yy} h_x - s_y h_{xy} + h_{xy} h_y)s_x)s_{xx} \\
&- 2(h_{yy} s_y^2 + 2(h_{yy} h_y + h_{xy} h_x)s_y)s_{xx} \\
&+ ((-6h_y - 4s_y)s_x - 10h_x s_y - 12h_y h_x)s_{xy} s_{yy} \\
&+ ((-4h_{xx} - 4h_{yy})s_y - 4h_{yy} h_y - 4h_{xy} h_x)s_x s_{xy} \\
&+ ((-12h_{yy} h_x + 12h_{xy} h_y)s_y - 8h_x h_{yy} h_y - 8h_x^2 h_{xy})s_{xy} \\
&+ (-4h_y^2 + s_x^2 + 2h_x s_x - 4s_y h_y - s_y^2)s_{yy}^2 \\
&+ ((4h_{yy} + 2h_{xx})s_x^2 + (-8h_{xy} h_y + 8h_{yy} h_x - 4s_y h_{xy})s_x)s_{yy} \\
&+ (-2h_{yy} s_y^2 + (-8h_{xy} h_x - 8h_{yy} h_y)s_y - 8h_x h_{xy} h_y - 8h_{yy} h_y^2)s_{yy} \\
&- s_x^6 - 6h_x s_x^5 - 3(4h_x^2 + 2s_y h_y + s_y^2)s_x^4 - 4(2h_x^3 + 6h_x s_y h_y + 3h_x s_y^2)s_x^3 \\
&- (3s_y^4 + 12h_y s_y^3 + 12(h_y^2 + h_x^2)s_y^2 - 2(-12h_x^2 h_y + h_{yyy} + h_{xxy})s_y)s_x^2 \\
&+ 4(h_{yyy} h_y + h_{yy} h_{xx} + h_{yy}^2 + h_{xy} h_x)s_x^2 \\
&- 6(h_x s_y^4 + 4h_x h_y s_y^3 + 4h_x h_y^2 s_y^2)s_x \\
&- 4(h_{yy} h_{xy} - h_x h_{yyy} + h_{xy} h_{xx} - h_{xxy} h_x)s_y s_x \\
&+ (8h_x h_{yyy} h_y + 8h_x^2 h_{xyy} - 8h_{xy} h_{yy} h_y + 8h_x h_{yy}^2)s_x - s_y^6 - 6h_y s_y^5 \\
&- 12h_y^2 s_y^4 + 2(h_{yyy} - 4h_y^3 + h_{xxy})s_y^3 + 4(h_{xxy} h_y + 2h_{yyy} h_y + h_{xyy} h_x)s_y^2 \\
&+ (-8h_x h_{xy} h_{yy} + 8h_{yyy} h_y^2 + 8h_x h_{xyy} h_y + 8h_{xy}^2 h_y)s_y = 0 \quad (22)
\end{aligned}$$

$$\begin{aligned}
& ((2h_y + s_y)s_x^2 + (4h_yh_x + 2h_xs_y)s_x + 4h_ys_y^2 + s_y^3 + 4h_y^2s_y)s_{xxx} \\
& + (-s_x^3 - 6s_x^2h_x + (-2s_yh_y - 8h_x^2 - s_y^2)s_x - 4h_xs_y^2 - 8h_xs_yh_y)s_{xxy} \\
& + ((s_y - 2h_y)s_x^2 + (-4h_yh_x + 2h_xs_y)s_x + s_y^3 - 4h_y^2s_y)s_{xyy} \\
& + (-s_x^3 - 2s_x^2h_x + (-s_y^2 - 2s_yh_y)s_x)s_{yyy} \\
& + ((-2s_y - 4h_y)s_x - 4h_yh_x - 2h_xs_y)s_{xx}^2 \\
& + (10h_xs_x - 6s_yh_y + 2s_x^2 + 8h_x^2 - 4h_y^2 - 2s_y^2)s_{xy}s_{xx} \\
& + (2s_xh_y + 2h_xs_y + 4h_yh_x)s_{yy}s_{xx} \\
& + (2h_{xy}s_x^2 + ((-2h_{xx} + 2h_{yy})s_y + 12h_{xy}h_x - 4h_{xx}h_y + 8h_{yy}h_y)s_x)s_{xx} \\
& + (-2h_{xy}s_y^2 + (4h_{yy}h_x - 4h_{xy}h_y)s_y + 8h_xh_{yy}h_y + 8h_x^2h_{xy})s_{xx} \\
& + 4(s_xh_y + 2h_yh_x + h_xs_y)s_{xy}^2 + 2(h_xs_x - s_y^2 + s_x^2 + 2h_y^2 + s_yh_y)s_{yy}s_{xy} \\
& + 4(h_{xy}h_y - h_{yy}h_x)s_xs_{xy} - 4(h_{xx} + h_{yy})s_y^2s_{xy} \\
& + 4(3h_{xy}h_x - 2h_{xx}h_y + h_{yy}h_y)s_ys_{xy} \\
& + (8h_xh_{xy}h_y + 8h_{yy}h_y^2)s_{xy} + (2h_y + 2s_y)s_xs_{yy}^2 \\
& + (2h_{xy}s_x^2 + ((6h_{yy} + 2h_{xx})s_y + 4h_{yy}h_y + 4h_{xx}h_y)s_x - 2h_{xy}s_y^2)s_{yy} \\
& + (-2h_{yyy} - 2h_{xxy})s_x^3 + (-4h_xh_{yyy} - 8h_{xxy}h_x - 4h_yh_{xyy})s_x^2 \\
& + ((-2h_{yyy} - 2h_{xxy})s_y^2 + (-4h_{xxy}h_y - 4h_{yyy}h_y + 4h_{yy}^2 + 4h_{yy}h_{xx})s_y)s_x \\
& + (-8h_xh_{xy}h_{yy} - 8h_xh_{xyy}h_y - 8h_x^2h_{xxy} + 8h_{yy}h_yh_{xx})s_x \\
& + (-4h_yh_{xyy} - 4h_{yy}h_{xy} - 4h_{xxy}h_x - 4h_{xy}h_{xx})s_y^2 \\
& + (-8h_y^2h_{xyy} - 8h_yh_xh_{xxy} + 8h_xh_{xy}^2 - 8h_{xx}h_yh_{xy})s_y = 0 \quad (23)
\end{aligned}$$

By the formula (4) with $\tilde{h} = h + s$ we obtain for the new Schrödinger potential

$$\tilde{u} = u - \Delta s + 2h_xs_x + s_x^2 + 2s_yh_y + s_y^2 \quad (24)$$

From the formula (20) we have for the new solution of the Fokker-Planck equation

$$\tilde{W} = e^{-s} \left(R_1 W - \frac{\partial W}{\partial y} + R_2 Q \right) \quad (25)$$

From the formulae (25), (13) and (3) we have for the new solution of the Schrödinger equation

$$\tilde{Y} = (R_1 + h_y) Y - \frac{\partial Y}{\partial y} + e^h R_2 Q \quad (26)$$

Where Q , according to (5), (6), (3) and (11), satisfies the following system of equations

$$\frac{\partial Y}{\partial x} Y_h - Y \frac{\partial Y_h}{\partial x} - \frac{\partial Q}{\partial y} = 0 \quad (27)$$

$$\frac{\partial Y}{\partial y} Y_h - Y \frac{\partial Y_h}{\partial y} + \frac{\partial Q}{\partial x} = 0 \quad (28)$$

The Darboux transformation (26) is a nonlocal transformation as the potential variable Q is a nonlocal variable connected with Y by the system (27), (28).

3 Application of the nonlocal Darboux transformation

In the simple case $h = 0$ the system of equations (22), (23) for $B = \exp(-s)$ has the form

$$\begin{aligned} & - (2 B B_y B_{xy} + B B_x (B_{xx} - B_{yy}) + B_x (B_x^2 + B_y^2)) (B_{xx} + B_{yy}) \\ & \quad + B (B_x^2 + B_y^2) \frac{\partial}{\partial x} (B_{xx} + B_{yy}) = 0 \\ & - (2 B B_x B_{xy} - B B_y (B_{xx} - B_{yy}) + B_y (B_x^2 + B_y^2)) (B_{xx} + B_{yy}) \\ & \quad + B (B_x^2 + B_y^2) \frac{\partial}{\partial y} (B_{xx} + B_{yy}) = 0 \end{aligned} \quad (29)$$

These equations were considered in the paper [8]. If B is a solution of equations (29) then $1/B$ and CB , where C is an arbitrary constant, are solutions as well. It is obvious from the form of equations (29) that any solution of the Laplace equation $\Delta B = 0$ is the solution of these equations.

The system of equations (29) can be integrated. In the case ΔB is not zero one can obtain

$$(B^2 - K) (B_{xx} + B_{yy}) - 2 B (B_x^2 + B_y^2) = 0 \quad (30)$$

or in the other form

$$\frac{B^4 \Delta (1/B)}{\Delta B} = -K \quad (31)$$

where K is an arbitrary constant. It is obvious that if B is a solution of equation (31) with constant K_B then $1/B$ is a solution of this equation with constant $1/K_B$.

According to equation (4), the initial potential u of the Schrödinger equation corresponding to $h = 0$ is $u = 0$. The new potential of Schrödinger equation corresponds to $s = -\ln(B)$ and is given by

$$\tilde{u} = \Delta B / B \quad (32)$$

Note that according to formula (32) B is an example of solution for the Schrödinger equation with new potential \tilde{u} .

The solution B_L of the Laplace equation provide $\tilde{u} = 0$ (32). Taking $B = 1/B_L$ one obtains nontrivial \tilde{u} , but potentials of this kind have singularities:

$$\tilde{u} = 2 (B_L)^{-2} \left(\left(\frac{\partial B_L}{\partial x} \right)^2 + \left(\frac{\partial B_L}{\partial y} \right)^2 \right)$$

Note that this formula coincides with the formula (12) for the Moutard transformation of the potential where $u = 0, Y_h = B_L$.

Let us consider the following ansatz for B

$$B(x, y) = F(x^2 + y^2)$$

This ansatz provides the following solution of equation (31)

$$B_r(x, y) = -\frac{\sqrt{K} \left((x^2 + y^2)^{C_1} - C_2 \right)}{(x^2 + y^2)^{C_1} + C_2} \quad (33)$$

where C_1, C_2 are arbitrary constants.

The solution B_r provides by the formula (32)

$$\tilde{u} = -8 \frac{C_2 C_1^2 (x^2 + y^2)^{C_1-1}}{\left((x^2 + y^2)^{C_1} + C_2 \right)^2} \quad (34)$$

This solvable potential is nonsingular if $C_1 \geq 1, C_2 > 0$. For $C_1 = 1, 2, 3, 4$ this potential is the special case of potentials considered in [8].

Consider for example $C_1 = 3/2$, $C_2 = 1$. In this case we have

$$B = \frac{(x^2 + y^2)^{3/2} - 1}{(x^2 + y^2)^{3/2} + 1} \quad (35)$$

Consider two solutions of the Laplace equation

$$Y_{L1} = \frac{x}{x^2 + y^2}, \quad Y_{L2} = \frac{y}{x^2 + y^2}$$

From (27), (28) with $Y_h = 1$ we have for $Y = Y_{L1}$ and $Y = Y_{L2}$

$$Q_{L1} = -\frac{y}{x^2 + y^2} + C_{L1}, \quad Q_{L2} = \frac{x}{x^2 + y^2} + C_{L2}$$

where C_{L1} , C_{L2} are arbitrary constants.

From (26) with $h = 0$ and B from (35) we have for $Y = Y_{L1}$, $Q = Q_{L1}$ and $Y = Y_{L2}$, $Q = Q_{L2}$

$$\tilde{Y}_{L1} = \frac{-2xy \left(7(x^2 + y^2)^{3/2} + 1 \right)}{(x^2 + y^2)^2 \left((x^2 + y^2)^{3/2} + 1 \right)} + \frac{C_{L1} x \left(5(x^2 + y^2)^{3/2} - 1 \right)}{(x^2 + y^2) \left((x^2 + y^2)^{3/2} + 1 \right)} \quad (36)$$

$$\tilde{Y}_{L2} = \frac{(x^2 - y^2) \left(7(x^2 + y^2)^{3/2} + 1 \right)}{(x^2 + y^2)^2 \left((x^2 + y^2)^{3/2} + 1 \right)} + \frac{C_{L2} x \left(5(x^2 + y^2)^{3/2} - 1 \right)}{(x^2 + y^2) \left((x^2 + y^2)^{3/2} + 1 \right)} \quad (37)$$

These functions are examples of solutions for the Schrödinger equation with solvable potential

$$\tilde{u} = -18 \frac{\sqrt{x^2 + y^2}}{\left((x^2 + y^2)^{3/2} + 1 \right)^2} \quad (38)$$

Now the Moutard transformation can be applied to the Schrödinger equation with potential (38). A single Moutard transformation can provide singular solvable potential. To obtain new nonsingular solvable potential for the Schrödinger equation from the solvable potential (38), let us apply twofold

Moutard transformation . Let us consider u from (38) and select two solutions Y_1, Y_2 of the Schrödinger equation equal to functions (36), (37) with $C_{L1} = C_{L2} = 0$. From the equation (18) we obtain

$$Q_{12} = \frac{49 (x^2 + y^2)^{3/2} + 1}{2 (x^2 + y^2)^2 \left((x^2 + y^2)^{3/2} + 1 \right)} + C/2 \quad (39)$$

where C is an arbitrary constant. From the equation (19) we obtain

$$\tilde{u} = \frac{-2r(441 + 9C^2r^8 + 2Cr(392r^6 + 49r^3 + 8))}{(Cr^4(r^3 + 1) + 49r^3 + 1)^2} \quad (40)$$

where $r = \sqrt{x^2 + y^2}$.

This solvable potential is nonsingular if $C \geq 0$. The example of solution for the Schrödinger equation with potential (40) is

$$\tilde{Y} = Y_1/Q_{12} = \frac{-4xy \left(7 (x^2 + y^2)^{3/2} + 1 \right)}{C (x^2 + y^2)^2 \left((x^2 + y^2)^{3/2} + 1 \right) + 49 (x^2 + y^2)^{3/2} + 1} \quad (41)$$

The solutions of equations (22), (23) can be obtained not only for $h = 0$. For example let us consider $h(x, y) = H(y)$ and use the ansatz $s(x, y) = -2H(y) + S(x)$. That reduces the equations (22), (23) to the ordinary differential equation

$$S_x S_{xxx} - S_{xx}^2 - S_x^4 = 0 \quad (42)$$

The solution of the equation (42) is

$$S(x) = \ln \left(\frac{\exp(C_1 x) - C_2}{\exp(C_1 x) + C_2} \right) + C_3 \quad (43)$$

where C_1, C_2, C_3 are arbitrary constants.

Thus from any solvable potential

$$u_H = -\frac{d^2 H(y)}{dy^2} + \left(\frac{dH(y)}{dy} \right)^2 \quad (44)$$

we obtain the new solvable potential

$$\tilde{u}_H = u_H + 2 \frac{d^2 H(y)}{dy^2} + \frac{2 C_2 C_1^2 \exp(C_1 x)}{(\exp(C_1 x) - C_2)^2} \quad (45)$$

Consider $H(y) = -\ln(\sin(y))$ and choose the solution (43) in the form $S(x) = \ln(\tanh(p(x - x_0)))$ where p, x_0 are arbitrary constants. In this case the initial solvable potential is $u_H = -1$ and

$$\tilde{u}_H = -1 + 2(\sin(y))^{-2} + 2p^2(\sinh(p(x - x_0)))^{-2} \quad (46)$$

This potential has singularities. To obtain new nonsingular solvable potential for the Schrödinger equation from the solvable potential (46), let us apply Moutard transformation. For the solution $\sin(x)$ of the Schrödinger equation with potential $u_H = -1$ by the formulae (27), (28) we obtain $Q = C - \cos(y)\cos(x)$ where C is an arbitrary constant. The formula (26) provides the solution of the Schrödinger equation with potential (46)

$$\tilde{Y}_p = \frac{(p(C - \cos(y)\cos(x)) - \cos(y)\sin(x)\tanh(p(x - x_0)))}{\sin(y)\tanh(p(x - x_0))} \quad (47)$$

We choose $Y_h = \tilde{Y}_p$ and perform the Moutard transformation for the potential (46). The new solvable potential is

$$-1 + \frac{2(f_1(\cosh(p(x - x_0)))^{-2} + f_2 + f_3\tanh(p(x - x_0)))}{(p(C - \cos(y)\cos(x)) - \cos(y)\sin(x)\tanh(p(x - x_0)))^2} \quad (48)$$

where

$$\begin{aligned} f_1 &= -p^2(p^2 + 1)(C - \cos(y)\cos(x))^2 + p^2C^2 - (p^2 + 1)\cos(y)^2 - \sin(x)^2 \\ f_2 &= -2p^2C\cos(y)\cos(x) + 1 + (p^2 - 1)\cos(x)^2 + (p^2 + 1)\cos(y)^2 \\ f_3 &= 2p\sin(x)(\cos(x) - C\cos(y)) \end{aligned}$$

This potential is obviously nonsingular for $p > 0$, $C > 1/p + 1$. The function $1/\tilde{Y}_p$ is an example of solution for the Schrödinger equation with potential (48).

4 Results and Discussion

In the past decades substantial progress in the description of the exact solutions for nonlinear partial differential equations and linear partial differential

equations with variable coefficients has taken place. The useful tool for solving one - dimensional Schrödinger equation is the Darboux transformation [11]. Some generalizations of the Darboux transformation for two - dimensional case were proposed, including operators of second order in derivatives [12], [13]. Examples of solvable two dimensional models were obtained in the framework of these generalized Darboux transformations (see for example [14], [15] and references therein). Some examples of exactly solvable two - dimensional stationary Schrödinger operators with smooth rational potentials decaying at infinity were obtained in the paper [10] by application of the Moutard transformation which is a two-dimensional generalization of the Darboux transformation. In the past years progress was made in the symmetry group analysis of differential equations by extending the spaces of symmetries of a given partial differential equations system to include nonlocal symmetries [16], [17]. In the present paper the inclusion of nonlocal variables in the Darboux transformation for the 2D stationary Schrödinger equation is considered.

In the paper [8] the Fokker-Planck equation associated with the two - dimensional stationary Schrödinger equation was considered. The Fokker-Planck equation has the conservation law form that yields a pair of potential equations. The special form of the Darboux transformation for these potential equations was introduced. As the potential variable is a nonlocal variable that provides the nonlocal Darboux transformation for the Schrödinger equation. It was shown that this nonlocal transformation is a useful tool for obtaining exactly solvable two - dimensional stationary Schrödinger operators. The consideration in the paper [8] is restricted by the simple case $\hbar = 0$. In the present paper the case of arbitrary \hbar is considered and relation of the nonlocal Darboux transformation with the Moutard [9] transformation is established. It is shown that the special case $\tilde{\hbar} = -\hbar$ of the nonlocal Darboux transformation provides the Moutard transformation. New examples of exactly solvable two - dimensional stationary Schrödinger operators with smooth potentials are obtained as an application of the nonlocal Darboux transformation.

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